Gas-ionizing shock and combustion waves in magnetogasdynamics

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Some general properties of one-dimensional deflagration waves in a non-conducting inviscid gas at rest are discussed when ionization of the gas takes place across a shock wave which precedes the flame front, and electromagnetic fields are present. The direction of wave propagation, the electric field and magnetic field are taken as a mutually orthogonal triad of vectors. The jump relationships across the gas-ionizing shock wave and magnetogasdynamic combustion wave are investigated and the two Hugoniot curves analysed in detail in the pressure-specific volume plane. The possible types of wave are indicated for arbitrary magnitudes of the upstream electromagnetic field. It is shown that weak gas-ionizing shock waves cannot exist. For suitably chosen electromagnetic field strengths the density ratio across the shock wave may be greater than the ordinary gasdynamic limit and, in such cases, the pressure and density ratios are related in an inverse manner, in contrast to the behaviour for ordinary gasdynamic or magnetogasdynamic shock waves. The magnetogasdynamic combustion wave has similar properties to that in ordinary gasdynamics.

1. Introduction

The properties of shock waves in gases with infinite electrical conductivity are well understood. The fundamental discontinuity relationships across the shock front have been examined and classified by Bazer & Ericson (1959) among others. The underlying structure which is the limit of a certain continuous fluid motion with finite values of viscosity, magnetic viscosity and heat conductivity has been investigated by Ludford (1959) and others. As a consequence of the infinite electrical conductivity unsupportable large values of the current develop on either side of the wave unless the electric and magnetic fields, **E** and **H**, are related by

$$\mathbf{E}+\mu\mathbf{v}\times\mathbf{H}=0, \tag{1}$$

where **v** is the fluid particle velocity and μ is the permeability. Here and in all subsequent equations m.k.s. Giorgi units of electromagnetic measurement are used.

However, if the temperature of the gas ahead of the shock wave is below a certain value it will not be ionized and its conductivity will be zero. For a monatomic gas the ionization temperature is approximately 10^4 deg K. Nevertheless, if a strong shock wave propagates into such a cold gas it is quite possible for the temperature to be raised above this value so that behind the wave the conductivity of the gas becomes high. In particular such phenomena may be expected to arise in deflagrations and detonations in which a large rise in temperature occurs across the shock wave preceding the flame front in a deflagration, or across the detonation front itself. In waves with a discontinuity of electrical conductivity across the front, which in an idealized case may be supposed to jump from zero ahead to infinitely large values behind, equation (1) must apply in the ionized region. Ahead of the wave, in view of the non-conductivity the current is zero in any case. Of course in actual practice no gas is either perfectly insulating or perfectly conducting and a theory based upon such idealizations may only be expected to have relevance when the magnetic Reynolds numbers ahead of and behind the front are respectively very much less than and greater than unity. The choice of that particular physical measurement to be taken as the typical length in the calculation of these magnetic Reynolds numbers is a problem of some complexity and is not discussed generally here. Such a relevant length might be, for instance, the distance over which sufficient ionization occurs for the gas to become highly conducting.†

The analysis of strong steady gas-ionizing shock waves was first reported by Liubimov (1959) and since that time further studies, chiefly by Russian scientists, have given considerable insight into the phenomena. It was shown by Liubimov & Kulikovsky (1960) that ahead of the ionizing shock wave there radiates a purely electromagnetic wave the character of which depends essentially upon the relative magnitudes of the dissipation coefficients and temperature in the upstream gas. Behind this wave the electromagnetic field is so adjusted as to permit the derivation of a unique solution, based upon considerations of the shock-wave structure, to unsteady problems such as, for example, the motion of a one-dimensional piston. In these papers the electromagnetic field was taken transverse to the wave front. A more general statement of the fundamental relations across a steady gas-ionizing shock wave with arbitrary orientation of the electromagnetic field ahead has been given by Zhilin (1960). Analogous relations have been derived by Helliwell & Pack (1962) for the general case of an unsteady curved shock or combustion wave. In this paper Zhilin goes on to show by means of two examples, that, even when the second law of thermodynamics is satisfied, such an arbitrarily specified electromagnetic field does not necessarily give rise to a steady shock wave. This result was already suggested by the work of Liubimov & Kulikovsky. In particular Zhilin demonstrates that this is so when the upstream electric field is zero as measured in a system of axes moving with the wave front. Thus, as he pointed out, the steady gas-ionizing shock wave originally studied by Liubimov (1959) has no physical existence.

In the present paper an analysis is made of the Hugoniot curves for certain types of gas-ionizing shock and combustion waves. These waves are examined in the context of a thermally non-conducting inviscid gas. Thus the very narrow transitional regions, in the flow of a real gas, within which steep gradients of the variables of state occur are replaced by surfaces across which these quantities are mathematically discontinuous. From a study of the discontinuities across such a surface, commonly termed a shock wave, information may be obtained

† I am indebted to a referee for this remark.

relating to the complete transition in the associated real-gas flow. The electromagnetic radiation wave ahead of the shock wave is not considered. The electric and magnetic fields behind this radiation wave are supposed mutually perpendicular and both orthogonal to the wave front, and may be of arbitrary magnitudes. In view of the remarks in the preceding paragraph some justification is required for this assumption. The analysis of Liubimov & Kulikovsky (1960) of the wave structure shows that steady shock waves may exist (in the sense that they are the limiting forms of some continuous flow) provided that the magnitudes of the electric and magnetic fields upstream are suitably related. The relationships are of three types depending upon the relative magnitudes of the kinematic viscosity, magnetic diffusivity and heat conductivity in the continuous flow. Two of these types lead to ordinary gas dynamic and magnetogasdynamic shocks, respectively. In the third type the relationship is of the form $\mathbf{E}+\mu\mathbf{v}_i\times\mathbf{H}=0$ where $\mathbf{v}_i$ is the velocity at which the temperature of the gas, within the transition region associated with the shock wave, attains the value for ionization. This case may clearly give rise to a wide range of values for E/H. Thus, in any particular instance, from the various possible Hugoniot adiabatic curves, which follow, should be selected that one with the relevant value as given by an analysis of the radiation wave.

2. A model deflagration and detonation

First we describe a one-dimensional model of a deflagration consisting of a combustion wave in which chemical energy is released and ahead of which a shock wave develops in the unburnt gas. Such a model would, for instance, be relevant to the problem of deflagration in a tube closed by a piston which moves into the gas. The release of energy at the flame front causes an expansion of the gas and a compressive wave is thus driven by the piston into the unburnt gas ahead of the combustion. The gas ahead of the shock wave is supposed at rest and non-conducting, $\sigma=0$ where σ is the conductivity. In this region mutually orthogonal electric and magnetic fields are assumed to exist, themselves normal to the direction of propagation of the waves. This shock is supposed sufficiently strong to ionize the gas so that behind it the conductivity $\sigma=\infty$. It is called a gas-ionizing shock wave. The flame front which propagates into the region behind the shock may thus be studied on the basis of pure magnetogasdynamics in a manner completely analogous to that of ordinary gasdynamics, since on both sides of it the gas is highly conducting, $\sigma=\infty$. The regions of gas, unburnt ahead of the shock wave, unburnt between the shock and combustion wave and burnt behind the combustion wave are referred to by the suffices 0, 1 and 2, respectively. The velocities of the shock and combustion fronts are written $\mathbf{V}^*$ and $\mathbf{W}^*$ in a system of axes fixed in the upstream gas with x-axis normal to the wave fronts and y-, z-axes along the directions of the upstream electric and magnetic fields, respectively. All other quantities measured relative to this system of axes are denoted by an asterisk. The particle velocity of the gas is indicated by the symbol $\mathbf{U}$. Thus $\mathbf{U}_r^*=U_r^*\mathbf{i}$ $(r=0,1,2)$; $\mathbf{E}_0^*=E_0^*\mathbf{j}$; $\mathbf{H}_0^*=H_0^*\mathbf{k}$. The situation is shown schematically in figure 1.

The model of a detonation wave is similar to that described above. Essentially the combustion and shock waves coalesce so that across a single front the conductivity increases from $\sigma = 0$ to $\sigma = \infty$. At the front there is a release of exothermal energy. The region 1 vanishes and $V^* = W^*$.

Across any discontinuity of the above type the jump relationships in the gas-dynamic and electromagnetic variables take the same general form. We use $[X]$ to denote the change in the value of X across the discontinuity. Variables without an asterisk are referred to axes relative to which the discontinuity is at

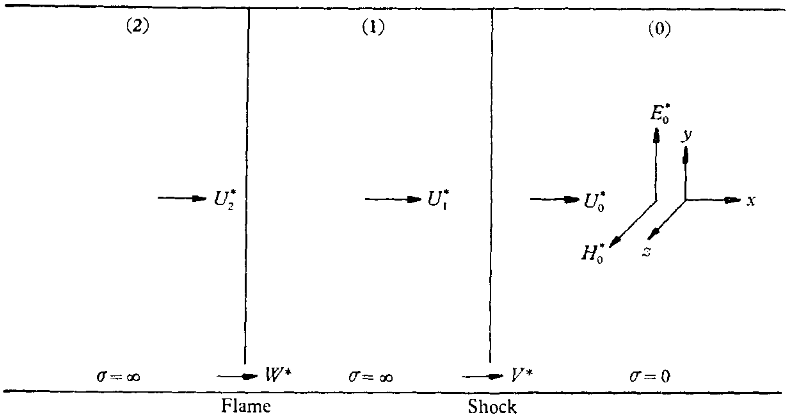

FIGURE 1. Model deflagration.

rest. Then (see, for example, Helliwell & Pack 1962) the various conservation equations lead to the following:

$$[H_n] = 0, \tag{2a}$$

$$[m] = [\rho U_n] = 0, \tag{2b}$$

$$[m\mathbf{U} + (p + \tfrac{1}{2}\mu H^2)\,\mathbf{n} - \mu H_n \mathbf{H}] = 0, \tag{2c}$$

$$[m(\tfrac{1}{2}U^2 + p/\rho + \mathscr{E}) + (\mathbf{E}\times\mathbf{H})_n] = mQ, \tag{2d}$$

$$[\mathbf{n}\times\mathbf{E}] = 0. \tag{2e}$$

Here $\mathbf{n}$ is the unit vector directed downstream normal to the wave front, m is the mass flow/unit area, p is the pressure, ρ is the density and $\mathscr{E}$ is the specific internal energy of the gas. The quantity Q is the exothermal energy per unit mass released at the flame front. In actual fact at the gas-ionizing shock front a certain energy of ionization is absorbed by the gas and $Q < 0$ may be used to represent this at the shock wave. However, the value is small and in this paper it is neglected. In addition to equations (2) the condition (1) holds in all regions where $\sigma = \infty$.

3. A gas-ionizing shock wave

A system of axes is chosen in which the shock front is at rest. Thus relative to an absolute system fixed in space the new system moves with a velocity $V^*\mathbf{i}$. Since V^* is small compared with the speed of light, the relativistic change in the electromagnetic fields yields $\mathbf{H}_r = \mathbf{H}_r^*$, $\mathbf{E}_r = \mathbf{E}_r^* + \mu(V^*\mathbf{i}\times\mathbf{H}_r^*)$ for $r = 0, 1$. Equation (1) leads to

$$\mathbf{E}_1 = -\mu U_1(H_{1z}\mathbf{j} - H_{1y}\mathbf{k}), \tag{3}$$

where subscripts x, y, z denote the respective component of $\mathbf{H}_1$. Now from equations (2e) we have

$$E_{1z} = 0.$$

Therefore $\mathbf{E}_1 = E_1\mathbf{j}$. It follows from equation (3), considered with equation (2a), that

$$H_{1y} = 0 = H_{1x}.$$

Thus $\mathbf{H}_1 = H_1\mathbf{k}$. The electric and magnetic fields throughout the flow therefore remain in their original orientations transverse to the wave front. It follows that the jump relations (2b–e) with equations (3) and no exothermal energy release become

$$m = U_1/\tau_1 = U_0/\tau_0, \tag{4a}$$

$$mU_1 + p_1 + \tfrac{1}{2}\mu H_1^2 = mU_0 + p_0 + \tfrac{1}{2}\mu H_0^2, \tag{4b}$$

$$m(\tfrac{1}{2}U_1^2 + p_1\tau_1 + \mathscr{E}_1) - E_1H_1 = m(\tfrac{1}{2}U_0^2 + p_0\tau_0 + \mathscr{E}_0) - E_0H_0, \tag{4c}$$

$$-\mu U_1 H_1 = E_1 = E_0. \tag{4d}$$

Here $\tau = 1/\rho$ is the specific volume of the gas. The relationships between the electromagnetic fields, pressure, density and velocity of the gas particles measured in the two systems of co-ordinates are

$$\mathbf{H}_r = H_r^*\mathbf{k}, \quad \mathbf{E}_r = (E_r^* - \mu V^* H_r^*)\mathbf{j}, \quad p_r = p_r^*, \quad \rho_r = \rho_r^*, \quad \mathbf{U}_r = -(V^* - U_r^*)\mathbf{i}, \tag{5}$$

for $r = 0, 1$. In particular if the upstream gas is at rest in the absolute system $\mathbf{U}_0 = -V^*\mathbf{i}$.

The modified Hugoniot curve

From equations (4b,c) we find

$$\tfrac{1}{2}(U_1+U_0)[(p_0-p_1)+\tfrac{1}{2}\mu(H_0^2-H_1^2)] + m[(p_1\tau_1 - p_0\tau_0) + (\mathscr{E}_1 - \mathscr{E}_0)] - (E_1H_1 - E_0H_0) = 0.$$

Thus, using equations (4a,d) to eliminate m and E_0, E_1, respectively, after a little algebra we obtain

$$\mathscr{E}_1 - \mathscr{E}_0 + \tfrac{1}{2}(\tau_1-\tau_0)(p_1+p_0) - \tfrac{1}{4}\mu(\tau_1+\tau_0)(H_1^2-H_0^2) + \mu\tau_1 H_1(H_1-H_0) = 0.$$

For a given equation of state which defines $\mathscr{E} = \mathscr{E}(p,\tau)$ this is the relationship between (p_1, τ_1) which is analogous to the Hugoniot adiabatic of ordinary gasdynamic shock-wave theory. In the following pages it is termed the modified Hugoniot curve.

A more convenient form is obtained if first we introduce a new variable, $\mathscr{H}$, such that

$$E_0 = \mu U_0 \mathscr{H}. \tag{6}$$

Then as a consequence of equations (4a,d) it follows that

$$H_1 = -\mathscr{H}(\tau_0/\tau_1). \tag{7}$$

It should be noted that when $\mathscr{H} + H_0 = 0$ the expression (6) relates the electric and magnetic fields upstream in the same manner as if the gas upstream had been perfectly conducting. Such shock waves would then be identical with what might be termed purely magnetogasdynamic shocks where the gas is highly conducting on both sides. However, in general, with H_1 replaced in terms of $\mathscr{H}$ by equation (7) the modified Hugoniot curve becomes

$$\mathscr{E}_1-\mathscr{E}_0+\tfrac{1}{2}(\tau_1-\tau_0)(p_1+p_0)-\tfrac{1}{4}\mu(\tau_1+\tau_0)[\mathscr{H}^2(\tau_0/\tau_1)^2-H_0^2] + \mu\tau_0\mathscr{H}(\mathscr{H}\tau_0/\tau_1+H_0) = 0. \tag{8}$$

The perfect gas

In what follows, to simplify the analysis and yet retain the essential features of the model, it is supposed that the gas is perfect throughout the entire flow but that the ratio of specific heats, $\gamma = c_p/c_v$, may take different constant values on either side of the front such that $\gamma_1 < \gamma_0$ (see, for example, Hayes & Probstein 1959). The range of values of γ for gases of physical interest is restricted to $1 < \gamma_1 < \gamma_0 < 2$. Then we have $\mathscr{E}_r = p_r\tau_r/(\gamma_r-1)$ for $r = 0, 1$. A further advantageous change of notation may now be made. Let us write the speed of sound upstream of the wave as $a_0 = (\gamma p_0\tau_0)^{\frac{1}{2}}$. Then we define in the upstream region non-dimensional Alfvén and pseudo-Alfvén speeds by

$$\alpha = H_0(\mu\tau_0)^{\frac{1}{2}}/a_0, \quad \beta = \mathscr{H}(\mu\tau_0)^{\frac{1}{2}}/a_0, \tag{9}$$

respectively. The case of a magnetogasdynamic shock wave is now represented by $\alpha+\beta = 0$. The case originally investigated by Liubimov (1959) is given by $\beta = 0$. Next introduce the further dimensionless speeds, u_r, such that $U_r = u_r a_0$ for $r = 0, 1$.

The insertion of the above notation into equations ($4a, b, c$) followed by a little algebraic manipulation leads to the following forms of the jump relations across the gas-ionizing shock wave:

$$u_1\tau_0 = u_0\tau_1, \tag{10a}$$

$$u_0(u_0-u_1) = \frac{1}{\gamma_0}\left(\frac{p_1}{p_0}-1\right)+\frac{1}{2}\left\{\beta^2\left(\frac{\tau_0}{\tau_1}\right)^2-\alpha^2\right\}, \tag{10b}$$

$$u_0^2-u_1^2 = \frac{2}{\gamma_0-1}\left(\frac{\gamma_1(\gamma_0-1)}{\gamma_0(\gamma_1-1)}\frac{p_1\tau_1}{p_0\tau_0}-1\right)+2\beta\left(\beta\frac{\tau_0}{\tau_1}+\alpha\right). \tag{10c}$$

In a similar manner the equation (8) for the modified Hugoniot curve is found to be

$$\frac{p_1}{p_0}\left(\frac{\gamma_1+1}{\gamma_1-1}-\frac{\tau_0}{\tau_1}\right) = \frac{\gamma_0\beta^2}{2}\left(\frac{\tau_0}{\tau_1}\right)^3-\frac{3\gamma_0\beta^2}{2}\left(\frac{\tau_0}{\tau_1}\right)^2 + \left\{\frac{\gamma_0+1}{\gamma_0-1}-\frac{\gamma_0\alpha^2}{2}\left(1+\frac{4\beta}{\alpha}\right)\right\}\left(\frac{\tau_0}{\tau_1}\right)-\left(1+\frac{\gamma_0\alpha^2}{2}\right). \tag{11}$$

We now investigate the properties of this curve. First we note that on it

$$\frac{p_1}{p_0} = \frac{\gamma_1-1}{\gamma_0-1}-\frac{\gamma_0(\gamma_1-1)}{2}(\alpha+\beta)^2, \quad \frac{\tau_1}{\tau_0} = 1. \tag{12}$$

Thus the curve passes below the point in the (p_1, τ_1)-plane corresponding to upstream conditions in contrast with the property of the corresponding curve for ordinary gasdynamic and magnetogasdynamic shocks. Hence it is not possible for a weak gas-ionizing shock wave to progagate without finite changes in pressure (or density) unless the electromagnetic field is itself an infinitesimal. The asymptotes of the curve (11) are

$$\frac{\tau_1}{\tau_0} \to \frac{\gamma_1 - 1}{\gamma_1 + 1} \quad \text{as} \quad \frac{p_1}{p_0} \to \infty, \tag{13a}$$

$$\frac{p_1}{p_0} \to -\left(\frac{\gamma_1 - 1}{\gamma_1 + 1}\right)\left(1 + \frac{\gamma_0 \alpha^2}{2}\right) \quad \text{as} \quad \frac{\tau_1}{\tau_0} \to \infty. \tag{13b}$$

The first of these is independent of (α, β) and thus identical with the corresponding ordinary gas dynamic asymptote. The second is, however, different, but is independent of β. In all the above respects the modified Hugoniot curve resembles the gasdynamic Hugoniot curve for combustion with an absorption instead of release of energy at the wave front.

Next we write equation (11) in the following form†

$$\frac{p_1}{p_0}\left(\frac{\gamma_1+1}{\gamma_1-1} - \frac{\tau_0}{\tau_1}\right) = \left(\frac{\tau_0}{\tau_1} - \frac{\gamma_1+1}{\gamma_1-1}\right)\left[\frac{\gamma_0\beta^2}{2}\frac{\tau_0}{\tau_1}\left\{\frac{\tau_0}{\tau_1} + \frac{2(2-\gamma_1)}{\gamma_1-1}\right\} + \frac{\gamma_1-1}{\gamma_1+1}\{1 + \tfrac{1}{2}\gamma_0\alpha^2 - F(\alpha,\beta)\}\right] - F(\alpha,\beta), \tag{14}$$

where

$$F(\alpha,\beta) = \frac{\gamma_0}{(\gamma_1-1)^3}\left[\{(\gamma_1-1)\alpha + (\gamma_1+1)\beta\}\{\gamma_1(\gamma_1-1)\alpha - (\gamma_1+1)(2-\gamma_1)\beta\} - \frac{2(\gamma_0+\gamma_1)(\gamma_1-1)^2}{\gamma_0(\gamma_0-1)}\right]. \tag{15}$$

Thus, when $F(\alpha,\beta) = 0$, the modified Hugoniot curves are

$$\text{(i)} \quad \frac{\tau_1}{\tau_0} = \frac{\gamma_1-1}{\gamma_1+1}, \tag{16}$$

$$\text{(ii)} \quad \frac{p_1}{p_0} = -\frac{\gamma_0\beta^2}{2}\frac{\tau_0}{\tau_1}\left\{\frac{\tau_0}{\tau_1} + \frac{2(2-\gamma_1)}{\gamma_1-1}\right\} - \frac{\gamma_1-1}{\gamma_1+1}\left\{1 + \frac{\gamma_0\alpha^2}{2}\right\}. \tag{17}$$

The latter gives negative values of the pressure ratio for all values of the density ratio which may be associated with real gases, viz. $\tau_1/\tau_0 \geqslant 0$. Although such pressure ratios are physically impossible we retain the theoretically possible formal values in order to describe the modified Hugoniot curves for which $F(\alpha,\beta) \neq 0$. In the (α,β)-plane it is observed that the curve for which $F(\alpha,\beta) = 0$ is a hyperbola with centre at the origin and asymptotes

$$\frac{\alpha}{\beta} = -\left(\frac{\gamma_1+1}{\gamma_1-1}\right), \quad \frac{\alpha}{\beta} = \frac{(\gamma_1+1)(2-\gamma_1)}{\gamma_1(\gamma_1-1)}.$$

It is shown in figure 2 for a typical case $1 < \gamma_0, \gamma_1 < 2$. It is seen that this curve divides the (α,β)-plane into two regions, interior and exterior to the hyperbola, labelled I and II in figure 2. For values of (α,β) in regions I, II it is easily verified that $F(\alpha,\beta) \lesseqgtr 0$, respectively.

We now proceed to discuss the general shape of the modified Hugoniot curves, using the singular curves (equations (16) and (17)) for which $F(\alpha,\beta) = 0$ and

† For this suggestion and consequent improvement upon my original analysis I am indebted to a referee.

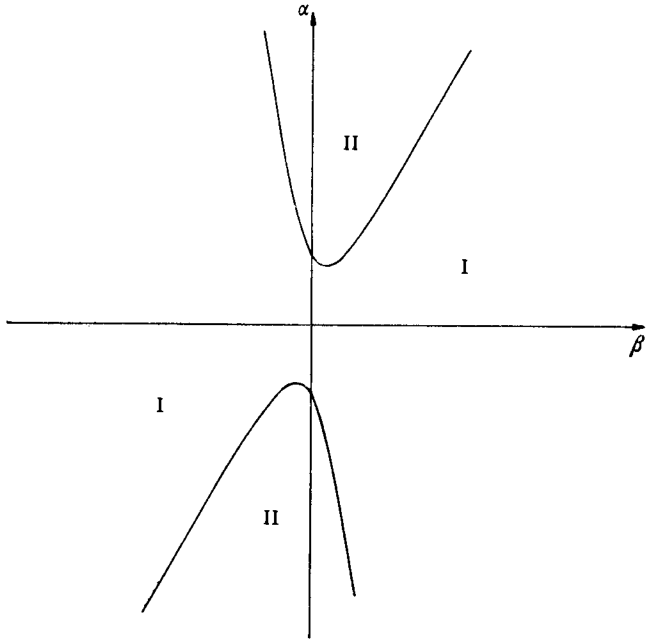

FIGURE 2. The (α, β)-plane.

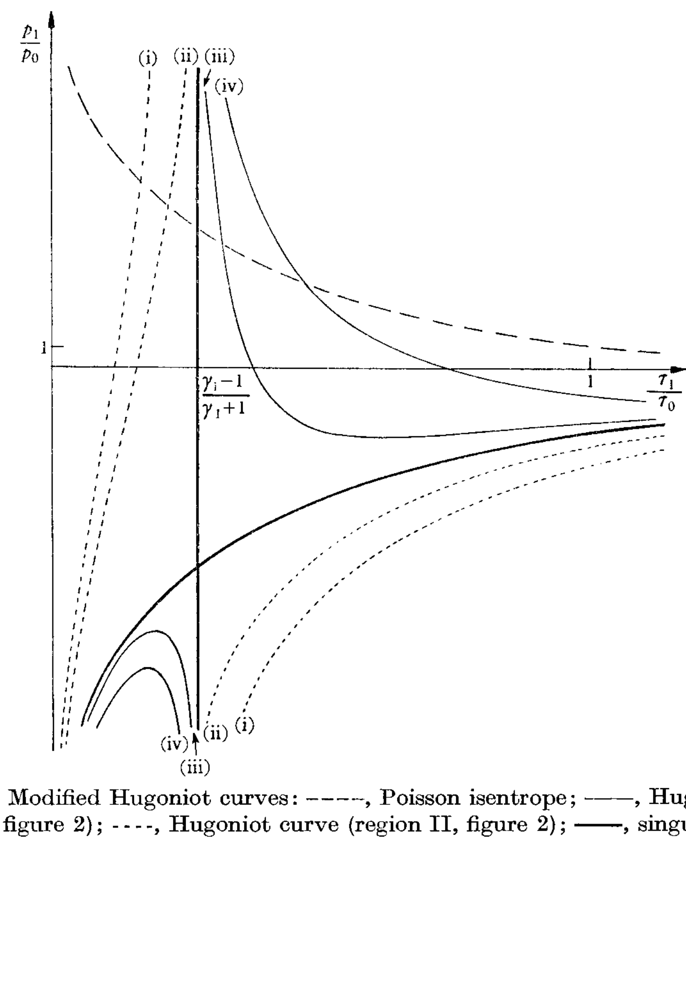

FIGURE 3. Modified Hugoniot curves: ––––, Poisson isentrope; ——, Hugoniot curve (region I, figure 2); - - - -, Hugoniot curve (region II, figure 2); ——, singular curves.

figure 2. The singular curves are shown with thick lines in figure 3. It is a simple matter to draw in the same figure the modified Hugoniot curves corresponding to $F(\alpha,\beta) \lesseqgtr 0$. First, we note that for a given value of p_1/p_0 the equation (14) for τ_1/τ_0 is a cubic. Now for $F(\alpha,\beta) < 0$ the modified Hugoniot curve (14) lies above the singular curve (17) when $\tau_1/\tau_0 > (\gamma_1-1)/(\gamma_1+1)$ and below when $0 < \tau_1/\tau_0 < (\gamma_1-1)/(\gamma_1+1)$. Thus it follows that for values of (α,β) in region I of figure 2 the modified Hugoniot curves ($\tau_1/\tau_0 > 0$) consist of the two branches shown with thinner unbroken lines in figure 3. The curves for values of (α,β) in region II of figure 2 similarly consist of the two branches shown with thinner broken lines in figure 3. A total of four such curves are shown in the figure for different values of $F(\alpha,\beta)$ and are labelled (i), (ii), (iii), (iv), respectively.

The entropy restriction

According to the second law of thermodynamics only those gas-ionizing shock waves may occur physically across which the entropy of the gas does not decrease on passing downstream. Thus among the general relations shown in figure 3 only those jumps are possible such that $s_1 \geqslant s_0$ where s is the specific entropy. For an isentropic change of state from the upstream conditions the relationship between the pressure p and specific volume τ is given by $p/p_0 = (\tau_0/\tau)^{\gamma_0}$. This is commonly termed the Poisson isentrope. For any change of state with increase of entropy the corresponding relationship is $p/p_0 > (\tau_0/\tau)^{\gamma_0}$. Thus, if the Poisson isentrope is superimposed upon figure 3, only those parts of the modified Hugoniot curves which lie above the Poisson isentrope may be associated with real ionization fronts. It is now clear, by reference to figure 3, that weak gas-ionizing shock waves cannot exist. The actual location of the point of intersection of the Poisson isentrope and modified Hugoniot curves cannot be determined explicitly. If it is supposed that at this point the specific volume is τ^* then, by equating the pressures on the two curves we find the value is given by the root ($0 < \tau^*/\tau_0 < 1$) of the equation

$$g\left(\frac{\tau_0}{\tau^*}\right) \equiv \left(\frac{\tau_0}{\tau^*}\right)^{\gamma_0+1} - \frac{\gamma_1+1}{\gamma_1-1}\left(\frac{\tau_0}{\tau^*}\right)^{\gamma_0} + \frac{\gamma_0+1}{\gamma_0-1}\left(\frac{\tau_0}{\tau^*}\right)$$
$$-1+\frac{\gamma_0}{2}\left[\beta^2\left(\frac{\tau_0}{\tau^*}\right)^3 - 3\beta^2\left(\frac{\tau_0}{\tau^*}\right)^2 - \alpha^2\left(1+\frac{4\beta}{\alpha}\right)\left(\frac{\tau_0}{\tau^*}\right) - \alpha^2\right] = 0. \qquad (18)$$

The mass flow restriction

An additional restriction upon the possible range of density ratio across a gas-ionizing shock front is given by the requirement that the mass flow through the wave must be real and positive (it may be zero in a nugatory case). From equations (10) it is straightforward to show that the mass flow, m, is given by

$$m^2\left(\frac{\tau_0}{a_0}\right)^2\left(1-\frac{\tau_1}{\tau_0}\right) = \frac{1}{\gamma_0}\left(\frac{p_1}{p_0}-1\right)+\frac{1}{2}\left[\beta^2\left(\frac{\tau_0}{\tau_1}\right)^2-\alpha^2\right].$$

Therefore, since $\tau_1/\tau_0 < 1$, for a real mass flow the condition is

$$\frac{p_1}{p_0} \geqslant 1-\frac{\gamma_0}{2}\left[\beta^2\left(\frac{\tau_0}{\tau_1}\right)^2-\alpha^2\right].$$

If the expression for p_1/p_0 in terms of τ_0/τ_1 from equation (11) is now inserted and the inequality rearranged it may be shown that the mass flow is real when

$$\frac{(\tau_0/\tau_1-\tau_0/\tau^{**})(\tau_0/\tau_1-\tau_0/\tau^{***})}{\tau_0/\tau_1-(\gamma_1+1)/(\gamma_1-1)} \leqslant 0,$$

where, respectively,

$$\frac{\tau_0}{\tau^{**}},\quad \frac{\tau_0}{\tau^{***}} = \frac{[(\gamma_1-1)\alpha\beta-(\gamma_1-1)/(\gamma_0-1)] \pm [\{(\gamma_1-1)\alpha\beta-(\gamma_1-1)/(\gamma_0-1)\}^2 + 2(\gamma_1/\gamma_0)(2-\gamma_1)\beta^2(1+\gamma_0\alpha^2/2)]^{\frac{1}{2}}}{(2-\gamma_1)\beta^2}. \tag{19}$$

For values of γ in the range $1 < \gamma_0, \gamma_1 < 2$, clearly $\tau_0/\tau^{***} < 0$. Hence the density ratio is restricted as follows:

$$\left.\begin{aligned} &\text{for } F(\alpha,\beta) < 0, \quad \text{region I (figure 2):} \\ &\qquad (\gamma_1-1)/(\gamma_1+1) < \tau_1/\tau_0 < \tau^{**}/\tau_0; \\ &\text{for } F(\alpha,\beta) > 0, \quad \text{region II (figure 2):} \\ &\qquad \tau^{**}/\tau_0 < \tau_1/\tau_0 < (\gamma_1-1)/(\gamma_1+1). \end{aligned}\right\} \tag{20}$$

These ranges exist only if $(\gamma_1-1)/(\gamma_1+1) \lesseqgtr \tau^{**}/\tau_0$ in the respective cases. By writing τ^{**}/τ_0 in terms of (α,β) from equation (19) and after some algebra it can be shown that this is so provided $F(\alpha,\beta) \lesseqgtr 0$, respectively, which is indeed the case. Thus the range of τ_1/τ_0 on the modified Hugoniot curve, already restricted from considerations of entropy, is restricted further under the following conditions:

$$\left.\begin{aligned} &\text{for } F(\alpha,\beta) < 0, \quad \text{region I (figure 2):} \\ &\qquad (\gamma_1-1)/(\gamma_1+1) < \tau_1/\tau_0 < \min(\tau^{*}/\tau_0, \tau^{**}/\tau_0); \\ &\text{for } F(\alpha,\beta) > 0, \quad \text{region II (figure 2):} \\ &\qquad \max(\tau^{*}/\tau_0, \tau^{**}/\tau_0) < \tau_1/\tau_0 < (\gamma_1-1)/(\gamma_1+1). \end{aligned}\right\} \tag{21}$$

It has not proved possible to analyse these conditions further for general values of (α,β). However, in the case analogous to an ordinary magnetogasdynamic shock in a perfectly conducting gas with $\gamma_1 = \gamma_0$ it is not difficult to show that the mass-flow requirement does not add any further restriction. In this case, when $\alpha+\beta = 0$, the appropriate domain of the (α,β)-plane is region I. Since at $\tau_1/\tau_0 = 1$, $p_1/p_0 = 1$ the entropy requirement is

$$(\gamma_0-1)/(\gamma_0+1) \leqslant \tau_1/\tau_0 \leqslant 1$$

and from equations (19) and (20) the mass-flow requirement is the same. Thus any magnetogasdynamic shock wave which satisfies the second law of thermodynamics is physically possible. In a similar manner the case $\gamma_1 = \gamma_0$ and $\beta = 0$ may be investigated. The inequalities corresponding to (20) are respectively $\tau_0/\tau^{**} = 1+\gamma\alpha^2/2 \lesseqgtr \tau_0/\tau_1 \lesseqgtr (\gamma_0+1)/(\gamma_0-1)$. If it is noted from equation (15) that $F(\alpha,\beta) = 0$ for $\gamma_0(\gamma_0-1)\alpha^2 = 4$, $\beta = 0$ we may deduce the following properties of the function $g(\tau_0/\tau^*)$ defined by equation (18). First, $g(1) < 0$. Secondly

$g(1+\gamma_0\alpha^2/2) \lesseqgtr 0$ according as $\gamma_0(\gamma_0-1)\alpha^2 \lesseqgtr 4$. Thus it follows that the root of equation (18) which defines the entropy restriction satisfies $\tau_0/\tau^* \gtrless \tau_0/\tau^{**}$ according as $\gamma_0(\gamma_0-1)\alpha^2 \lesseqgtr 4$, that is according as (α, β) lies in region I or II of the (α, β)-plane, respectively. Hence, in this case also, mass-flow considerations introduce no further restrictions. Indeed in several other cases investigated numerically the condition of real mass flow is found to yield no additional restriction, and it is conjectured that this may hold generally. Nevertheless, in the absence of a complete analysis, for any particular values of (α, β) the results of this paragraph should be taken into account and inequalities (21) used to define the possible range of density jumps across the fronts.

Summary of results

Upstream of the gas-ionizing shock wave, which moves into a perfect gas at rest, the electromagnetic field is denoted by the parameters (α, β) defined by equations (6) and (9). The case $\alpha+\beta = 0$ is analogous to a magnetogasdynamic shock wave in a perfectly conducting gas. The case $\beta = 0$ corresponds to a flow with zero upstream electric field relative to an observer moving with the shock wave itself. The behaviour of the discontinuity is essentially different for differing (α, β) depending upon the domain of the (α, β)-plane in which any particular values lie, as shown in figure 2. The modified Hugoniot curves corresponding to the various regions are shown in figure 3.

For (α, β) in region I, the general behaviour is similar to that in ordinary gasdynamics, with the following differences. At $\tau_1/\tau_0 = 1$, $p_1/p_0 \leqslant 1$ with equality only in the case $\alpha+\beta = 0$ with $\gamma_1 = \gamma_0$. For any other values of (α, β) there is an upper limit (< 1) to the density ratio, and a lower limit (> 1) to the pressure ratio across the shock. These limits are given by inequality (21). The overall maximum possible density ratio is $(\gamma_1+1)/(\gamma_1-1)$ as in ordinary and magnetogasdynamics.

For (α, β) in region II the behaviour is markedly different from that in ordinary gasdynamics. In such cases the density ratio is always greater than

$$(\gamma_1+1)/(\gamma_1-1),$$

the minimum possible value being given by the inequality (21). Further, in contrast with the previous case, the pressure and density ratios across the gas-ionizing shock front are related inversely so that a larger pressure ratio is associated with a smaller density ratio, and conversely. Such a relationship, first noted by Liubimov (1959) is completely different from that for any other known type of shock wave.

4. A magnetogasdynamic combustion wave

The analysis of the combustion wave is similar to that already carried through for the shock wave behind which the electrical conductivity of the gas is supposed infinite. Thus the combustion wave propagates into a fully ionized gas and we shall refer to it as a magnetogasdynamic combustion wave. In this case the relationship (1) holds on both sides of the wave. As indicated in figure 1 the variables ahead and behind the wave front are denoted respectively by the suffices 1 and 2.

A system of axes is chosen moving with the combustion front, that is the velocity of the new set relative to the absolute set of axes is $W^*\mathbf{i}$. Then, referred to the new axes the components of the electric fields are given by

$$E_r = E_r^* - \mu W^* H_r,$$

and the particle speeds by $U_r = W^* - U_r^*$ for $r = 1, 2$. It should be noted that for these variables $\phi_r \equiv U_r, E_r$ in the present section the interpretation of ϕ_1 is in general different from that in the previous section. These should be no confusion however. For the gas-ionizing shock front $\phi_1 = \text{fn}\,(V^*, \phi_1^*)$ and for the combusion front $\phi_1 = \text{fn}\,(W^*, \phi_1^*)$, where the interpretation of ϕ_1^* is common, namely, the value referred to an absolute system of axes. It then follows exactly as before that there is no rotation of the electric or magnetic fields across the wave front. Equations (5) may now be used to relate the values of the several variables defined in the absolute and moving systems of axes provided that we replace the suffices $(0, 1)$ by $(1, 2)$ and write W^* for V^* throughout. Thus, when exothermal energy of amount Q per unit mass is released at the flame front the inclusion of the condition (1) into the jump relations (2) leads to

$$m = U_2/\tau_2 = U_1/\tau_1, \tag{22a}$$

$$mU_2 + p_2 + \tfrac{1}{2}\mu H_2^2 = mU_1 + p_1 + \tfrac{1}{2}\mu H_1^2, \tag{22b}$$

$$m(\tfrac{1}{2}U_2^2 + p_2\tau_2 + \mathscr{E}_2) - E_2 H_2 = m(\tfrac{1}{2}U_1^2 + p_1\tau_1 + \mathscr{E}_1) - E_1 H_1 + mQ, \tag{22c}$$

$$-\mu U_2 H_2 = E_2 = E_1 = -\mu U_1 H_1. \tag{22d}$$

From equations $(22a, d)$ follows the well-known magnetogasdynamic property that the magnetic field is 'frozen' into the fluid, and

$$H_2/H_1 = U_1/U_2 = \tau_1/\tau_2. \tag{23}$$

The modified Hugoniot curve for combustion then follows from equations $(22b, c)$ and upon elimination of H_2 by means of equation (23), becomes

$$\mathscr{E}_2 - \mathscr{E}_1 + \tfrac{1}{2}(\tau_2 - \tau_1)(p_2 + p_1) - \tfrac{1}{4}\mu H_1^2(\tau_2 + \tau_1)[(\tau_1/\tau_2)^2 - 1] + \mu\tau_1 H_1^2[(\tau_1/\tau_2) - 1] = Q. \tag{24}$$

Consider the jump relations (22). If we write

$$p_r' = p_r + \tfrac{1}{2}\mu H_r^2, \quad \mathscr{E}_r' = \mathscr{E}_r + \tfrac{1}{2}\mu H_r^2 \tau_r, \tag{25}$$

for $r = 1, 2$ they become

$$m = U_2/\tau_2 = U_1/\tau_1,$$

$$m(U_2 - U_1) + (p_2' - p_1') = 0,$$

$$\tfrac{1}{2}(U_2^2 - U_1^2) + (\mathscr{E}_2' + p_2'\tau_2) - (\mathscr{E}_1' + p_1'\tau_1) = Q.$$

These are identical with the standard equations across a flame front in ordinary gasdynamics for a fictitious gas whose equation of state is defined by equations (25). The properties of such waves may thus be determined by a suitable application of ordinary flame analysis, see, for example, Courant & Friedrichs (1949).

For instance, a steady-stage detonation wave is associated with the Chapman–Jouguet point on the modified Hugoniot curve for combustion, where

$$-\rho_2^2 c_2^2 = dp_2'/d\tau_2 = (p_2'-p_1')/(\tau_2-\tau_1) = -\rho_2^2 U_2^2,$$

and c_2 is the speed of propagation of small disturbances behind the flame front in this fictitious gas. But corresponding to the conservation equation $H_2\tau_2 = H_1\tau_1$ we have the differential relationship $d(H\tau) = 0$. Thus

$$c_2^2 = -(1/\rho_2^2)\,dp_2'/d\tau_2 = dp_2/d\rho_2 + \mu H_2^2/\rho_2 = a_2^2 + b_2^2,$$

where b_2 is the Alfvén speed behind the flame front. We have therefore deduced the result obtained previously by Gross, Chinitz & Rivlin (1960) that in a steady-state magnetogasdynamic detonation wave the wave front moves with 'magnetosonic' speed relative to the products of combustion. Whilst similar results could be obtained by similar use of the fictitious gas, nevertheless, it is instructive to analyse explicitly the properties of the Hugoniot curve.

The perfect gas

Although the burning of the gas at the flame front may be expected to change its thermodynamic properties, for simplicity it is supposed that the gas remains perfect but the ratio of specific heats, γ, suffers a discontinuity at the wave front. It is known that, in general, $1 < \gamma_2 < \gamma_1 < 2$. Then the specific internal energy $\mathscr{E}_r = p_r\tau_r/(\gamma_r-1)$ for $r = 1, 2$. At this stage it is convenient to introduce a dimensionless notation in terms of the speed of sound upstream, $a_1 = (\gamma_1 p_1 \tau_1)^{\frac{1}{2}}$, as unit of measure where $a_1/a_0 = \{(\gamma_1/\gamma_0)(p_1/p_0)(\tau_1/\tau_0)\}^{\frac{1}{2}}$. Thus, upstream of the front the Alfvén speed is

$$\delta = |H_1(\mu\tau_1)^{\frac{1}{2}}/a_1| = |-\beta(\tau_0/\tau_1)(p_0/p_1)^{\frac{1}{2}}|. \tag{26}$$

Also set $W^* = w^* a_1$, $U_r = u_r a_1$ for $r = 1, 2$ and $Q = qa_1^2$. In terms of these quantities the jump relations (22) across the front become

$$u_2\tau_1 = u_1\tau_2, \tag{27a}$$

$$u_1(u_1-u_2) = \frac{1}{\gamma_1}\left(\frac{p_2}{p_1}-1\right) + \frac{\delta^2}{2}\left\{\left(\frac{\tau_1}{\tau_2}\right)^2 - 1\right\}, \tag{27b}$$

$$u_1^2-u_2^2 = \frac{2}{\gamma_1-1}\left\{\frac{\gamma_2(\gamma_1-1)}{\gamma_1(\gamma_2-1)}\left(\frac{p_2\tau_2}{p_1\tau_1}\right)-1\right\} + 2\delta^2\left\{\frac{\tau_1}{\tau_2}-1\right\} - 2q. \tag{27c}$$

The modified Hugoniot curve (24) is

$$\frac{p_2}{p_1}\left[\frac{\gamma_2+1}{\gamma_2-1}-\frac{\tau_1}{\tau_2}\right] = \frac{\gamma_1}{2}\delta^2\left(\frac{\tau_1}{\tau_2}\right)^3 - \frac{3\gamma_1\delta^2}{2}\left(\frac{\tau_1}{\tau_2}\right)^2 + \left\{\frac{\gamma_1+1}{\gamma_1-1}+\frac{3\gamma_1\delta^2}{2}+2\gamma_1 q\right\}\left(\frac{\tau_1}{\tau_2}\right) - \left(1+\frac{\gamma_1\delta^2}{2}\right). \tag{28}$$

A comparison with the modified Hugoniot curve given by equation (11) confirms that the equation of the modified Hugoniot curve for combustion has a form appropriate to the magnetogasdynamic case $\alpha+\beta = 0$ with the addition of an exothermal energy q. The form of this modified Hugoniot curve is that shown in figure 3 in the special case when $p_1/p_0 = (\gamma_1-1)/(\gamma_0-1)$ at $\tau_1/\tau_0 = 1$. We proceed to show that, in the magnetogasdynamic case as in ordinary gasdynamics, this curve is of a similar nature to the modified Hugoniot curve without combustion.

First it is noted from equation (28) that

$$\frac{p_2}{p_1} = \frac{\gamma_2 - 1}{\gamma_1 - 1} + \gamma_1(\gamma_2 - 1)\,q, \quad \text{at} \quad \frac{\tau_2}{\tau_1} = 1. \tag{29}$$

Therefore, exactly as in ordinary gasdynamics, there is in general an excess pressure associated with combustion at constant volume; its value is independent of δ, the magnetic field parameter. The asymptotes of the adiabatic (28) are

$$\frac{\tau_2}{\tau_1} \to \frac{\gamma_2 - 1}{\gamma_2 + 1}, \quad \text{as} \quad \frac{p_2}{p_1} \to \infty, \tag{30a}$$

$$\frac{p_2}{p_1} \to -\left(\frac{\gamma_2 - 1}{\gamma_2 + 1}\right)\left(1 + \frac{\gamma_1 \delta^2}{2}\right), \quad \text{as} \quad \frac{\tau_2}{\tau_1} \to \infty. \tag{30b}$$

If $\gamma_2 = \gamma_1$ the first of these is the same as in ordinary gasdynamics. Both are independent of the exothermal energy release and identical with the corresponding asymptotes on the general ionization adiabatics.

Now let us write equation (28) in the form

$$\frac{p_2}{p_1}\left[\frac{\gamma_2+1}{\gamma_2-1} - \frac{\tau_1}{\tau_2}\right] = \left[\frac{\tau_1}{\tau_2} - \frac{\gamma_2+1}{\gamma_2-1}\right]\left\{\frac{\gamma_1\delta^2}{2}\frac{\tau_1}{\tau_2}\left[\frac{\tau_1}{\tau_2} + \frac{2(2-\gamma_2)}{\gamma_2-1}\right] + \frac{\gamma_2-1}{\gamma_2+1}\left[1 + \frac{\gamma_1\delta^2}{2} - G(\delta)\right]\right\} - G(\delta), \tag{31}$$

where

$$G(\delta) = -\frac{1}{\gamma_2 - 1}\left\{\frac{4\gamma_1\delta^2}{(\gamma_2-1)^2} + \frac{2(\gamma_1+\gamma_2)}{\gamma_1 - 1} + 2\gamma_1(\gamma_2+1)\,q\right\}. \tag{32}$$

It is clear that the general shape of the Hugoniot curve represented by this equation may be obtained immediately by analogy with the earlier discussion following equations (14), (15) for the corresponding case of gas-ionizing shock waves. The singular Hugoniot curves when $G(\delta) = 0$ are identical with equations (16), (17) in which α^2, β^2 are both replaced by δ^2 and suffices $(0, 1)$ by $(1, 2)$, respectively. Furthermore, for all physically realistic combustion waves $G(\delta) < 0$. Thus the general shape of the modified Hugoniot curve for combustion is that shown in figure 4 for values of $\tau_2/\tau_1 > 0$.

For comparative purposes the ordinary gasdynamic Hugoniot curve for combustion, for the same value of the exothermal energy and the same change of adiabatic index, is also shown on figure 4. The two curves have the same slope at the common point corresponding to combustion at constant volume. For combustion giving rise to an increase (decrease) of density the magnetogasdynamic wave is associated with a greater (lesser) increase of pressure than occurs for the ordinary gasdynamic wave. In addition the Poisson isentrope is also shown on figure 4. The condition of entropy increase across the wave front then leads to the result that in magnetogasdynamics the maximum possible rarefaction across a combustion wave is less than that in the corresponding ordinary gasdynamic case. Finally, it can be shown, by an analysis similar to that for a gas-ionizing shock wave, that the mass flow is real across any combustion wave which satisfies the second law of thermodynamics.

The concepts of weak and strong deflagrations call for examination in the light of the effects already noted above as a consequence of the interaction of the gasdynamic and electromagnetic energies. However, such a study is beyond the scope of the present paper.

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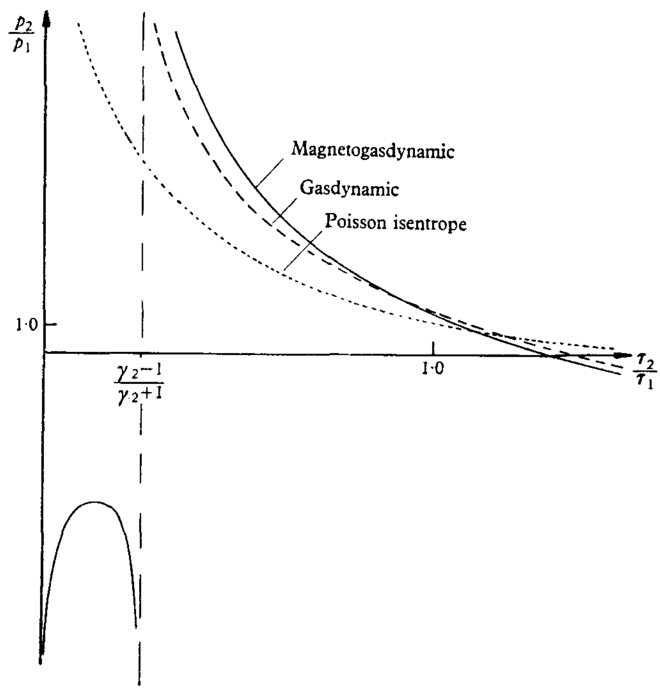

FIGURE 4. Hugoniot curves for combustion.

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